Technical Comments

Solving Problems of Elasto-Plastic Flow

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IN Ref. 1, Akyuz and Merwin document a solution technique for a class of problems of increasing engineering interest. It would appear from the presentation of their work that two comments are in order. First, the authors have failed to recognize the essential mathematical nature of their problem in that they refer to it as nonlinear. Second, many of the procedural aspects of their work have been predated.

That the problem is not nonlinear may be deduced by observing carefully the structure of the governing equations. In addition, the basic relations imply the same information. The authors recognize the linear relation between strain and stress increments which, in compact notation, takes the form

$$d\epsilon_{ij} = {}_{ijkl}d\sigma_{kl}, \quad \mathfrak{D}i,j,k,l = 1,2,3 \tag{1}$$

In (1), $d\epsilon_{ij}$ and $d\sigma_{kl}$ are these increments; $\mathfrak{D}_{ijkl}$ is a compliance that depends on the instantaneous values of the stresses and the stress-strain relation the material obeys.

If (1) is combined with a) equilibrium equations in terms of the stress increments and b) strain-displacement equations also in terms of increments, a complete problem in terms of incremental quantities may be defined. This problem has several interesting features. It requires both boundary and initial conditions. It is neither linear nor nonlinear (in terms of incremental quantities), but it is quasi-linear; that is, the dependent variables occur in a manner such that their highest derivatives are linear. Thus, although the physical event remains nonlinear, the mathematical problem is quasi-linear.

If, in addition, the stress-strain curve is presumed to be monotonic, with a continously turning tangent—as Akyuz and Merwin have done—an additional feature accrues. The governing equations may be shown to be elliptic. In this case, there are no slip lines or discontinuities of any sort in the field. This feature has enormous implications for numerical analysis of which Akyuz and Merwin take advantage. During the initial load increment, when the response is wholly elastic, the behavior is, of course, elliptic. The same behavior is guaranteed for all subsequent load increments so that they never need be concerned with the transition to slip line generation. A more detailed statement and derivation of these characteristics of the problem appear elsewhere.[2]

That some of the procedures have been worked out previously may be seen by consulting the literature; see, e.g., Refs. 3–8.

References

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Reply by Author to J. L. Swedlow

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THE following two paragraphs are written in the hope of clarifying the confusion that arose in the mind of the author of Ref. 1 and corresponding to the two comments mentioned therein.

1) The term "nonlinear" in the title of Ref. 2 refers (as is clear from the reading and understanding of Ref. 2) to the facts that the geometry of the boundary and the prevailing conditions therein are nonlinear functions of the unknown variables and that the stiffness due to the initial stresses and incremental rotations is taken into account. Since these are prominent aspects of the paper, the authors deliberately included the term nonlinear in the title. In addition to these obvious reasons for the term nonlinear one can read on page 1827 of Ref. 2, following Eq. (19), that "Equations (16) and (17) constitute a set of nonlinear equations for the solution of $d\sigma_{zz}$," which means that in the plane strain case Eq. (1) of Ref. 1 becomes mathematically nonlinear, exactly in the sense the author in Ref. 1 would like to see it.

2) At the bottom of page 1825 of Ref. 2, a note "Presented as Paper 67-144 at the AIAA 5th Aerospace Sciences Meeting, New York, January 23–26, 1967" eliminates the possibilities of Ref. 2 being predated by the list of Refs. 1–4 and 8 cited in Ref. 1. Actually, as far as procedural aspects of these works are concerned, Ref. 3 presents clearly the derivation of Prandtl-Reuss equations for both perfectly plastic material and for material with strain hardening. Reference 4 treats the incremental load technique, Ref. 5 treats the application of finite element and direct stiffness method with incremental load technique, and Ref. 6 indicates the application of Prandtl-Reuss equations to the finite-element and force method. The authors regret having missed any additional information in Refs. 5–7 of Ref. 1 during the preparation of their paper.

As explained in the foregoing comment 1) the intrinsic nature of the problem treated in Ref. 2 is totally different from the ones treated in Refs. 1–8 of Ref. 1 and Refs. 4–6 of this article. Furthermore, the complexity of the problem in Ref. 2, in contrast to the simplicity by which the problems could be treated in the above references, required a strong positive definite behavior of the stiffness matrix at each in-

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cremental indentation step. This fact has been stated in the first paragraph of the introduction of Ref. 2 with a typical stress-strain curve for material with strain hardening and the implications that arise from the relative amount of incremental loading and the slope of the stress-strain curve, therefore, the degree of positive definiteness of the stiffness matrix obtained has been explained with reference to Fig. 2; and one method of diagnosis for ill behavior of the stiffness matrix has been explained with reference to Fig. 3 on page 1827 of Ref. 2.

References

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A Further Note on Laminar Incipient Separation

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Nomenclature

H = boundary-layer form factor
K = pressure gradient parameter
M = Mach number
S_w = total enthalpy function
T = temperature
τ = shear stress
θ_i = incipient compression surface deflection angle
$\bar{\chi}$ = viscous interaction parameter

Subscripts

o = flat plate, i.e., $K_o = 0$
s = separation, i.e., $\tau_w = 0$
tr = transformed
w = wall

REFERENCE 1 presented the effect of wall temperature on the incipient deflection angle, θ_i, i.e., that angle which the laminar boundary layer can negotiate without separating. The wall temperature was shown to enter into the valuation of λ in the equation

$$M\theta_i = \lambda\bar{\chi}^{1/2} \tag{1}$$

where

$$\lambda = -2.45K_s\,(-\Delta H_{\rm tr}/\Delta K)^{1/2} \tag{2}$$

Although λ may be evaluated accurately through Eq. (2) by making use of the similar solutions of Ref. 2, its physical significance was not apparent.

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Recalling that

$$\Delta H_{\rm tr} \equiv H_{{\rm tr}_s} - H_{{\rm tr}_o}$$
$$\Delta K \equiv K_0 - K_s = -K_s,\ K_o = 0$$

and noting that, very accurately, independent of the wall-to-stagnation temperature ratio,

$$K_sH_{{\rm tr}_s} \cong -0.275,\ -1 \leq S_w \leq 1$$

one obtains, upon substitution into Eq. (2),

$$\lambda = 1.288(1 - H_{{\rm tr}_o}/H_{{\rm tr}_s})^{1/2} \tag{3}$$

Therefore, λ is a function of the ratio of the transformed form factor at the beginning of the interaction and at the point of incipient separation. Finally, it is noted that the ratio $H_{{\rm tr}_o}/H_{{\rm tr}_s}$ increases nonlinearly with S_w only through $H_{{\rm tr}_s}$ since $H_{{\rm tr}_o} = 2.591\ (S_w + 1)$, $H_{{\rm tr}_s}$ being inversely proportional to the pressure gradient at separation.

References

[1] Ball, K. O. W., "Wall Temperature Effect on Incipient Separation," *AIAA Journal*, Vol. 5, No. 12, Dec. 1967, pp. 2283–2285.

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Comment on "Exact Solution of Certain Problems by Finite Element Method"

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IN his recent Note, Tong[1] proved that when the exact solution to the homogeneous Euler Equations of a positive definite functional[2] with one independent variable is known, and it is used as interpolation functions in the variational formulation of the finite element equations, the generalized displacements which are the solution to these equations constitute the exact solution to the problem at the nodal points regardless of the number or size of the elements used in the discretization. This property is often used in many one-dimensional problems and also in two-dimensions when a separation of variables is applicable. Typical problems of this kind are those concerning continuous beams, frame structures, some plate problems, and axisymmetric shells of revolution.

It is of interest to note, however, that an alternate derivation of the finite element equations has been used by engineers before the advent of the finite element method as such. If the generalized displacements are defined as the displacements at the element ends, the Euler Equations of the appropriate functional are equilibrium equations at the same points. In Tong's Note these are given in matrix form in Eq. (8). Obviously the entries of the stiffness matrix **K** are the end forces corresponding to unit values of the end displacements, and they can be obtained from the general solution to the homogeneous equations. Thus, when these end forces are obtained from the exact general solution, they are exact too.

Then, it remains to prove that the generalized forces in Eq. (8), **Q**, are also exact. The generalized forces are defined

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